AIAA 79-1669R

Instability of Controlled Projectiles in Ascending or Descending Flight

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Lloyd and Brown have shown that constant horizontal and vertical side forces and moments applied to a spinning projectile can result in dynamic instability. This instability arises from the nonlinear terms in the fixed-plane coordinate system spin that appear in the equations of motion. By the use of the fixed-plane system (rather than Lloyd and Brown's nonrolling system) simpler relations can be developed, the full effects of gravity, drag and roll damping obtained, and the limitation to large gyroscopic stability factors removed. Very simple stability bounds are given for spinning finned missiles with spin rates well above resonance as well as for a gyroscopically stable shell. Exact solutions are given for finned missiles with zero spin but the motion of slowly spinning missiles should be numerically computed.

Nomenclature

B_1, B_2 = lower and upper bounds on $\hat{\beta}_e \tan\theta_e$
C = that part of the fixed-plane complex yaw forcing function due to the control force and moment
C_D = (drag force)/$\frac{1}{2}\rho SV^2$
C_{L_α} = (lift force)/$\frac{1}{2}\rho SV^2 |\xi|$
C_{l_p} = roll damping moment coefficient
C_{l_δ} = roll moment coefficient due to canted fins
$C_{M_{p\alpha}}$ = (Magnus moment)/$\frac{1}{2}\rho SlV^2\phi' |\xi|$
$C_{M_q}+C_{m_{\dot\alpha}}$ = (sum of the damping moments)/$\frac{1}{2}\rho SlV^2 |\mu|$
C_{M_α} = (static moment)/$\frac{1}{2}\rho SlV^2 |\xi|$
C_{N_α} = normal force coefficient $=\gamma C_{L_\alpha}+C_D$
D_1 $= C_D^* + k_a^{-2} C_{l_p}^*$
D_2 $= k_a^{-2}\delta_f C_{l_\delta}^*$
E $= \frac{1}{2}\hat{\xi}_e \tan\theta_e$
F_Y, F_Z = transverse missile-fixed components of the aerodynamic force
F_{YC}, F_{ZC} = transverse fixed-plane components of the control force
G $= PglV^{-2}\cos\theta_e$
$\hat{G}$ = that part of the fixed-plane complex yaw forcing function due to gravity
g = magnitude of the gravity acceleration
g^* $= glV^{-2}\sin\theta_T$
H $=\gamma C_{L_\alpha}^* - C_D^* - k_t^{-2}(C_{M_q}^* + \gamma C_{M_{\dot\alpha}}^*)$
I_x, I_y = axial and transverse moments of inertia
K_j = magnitude of the jth modal arm, $j=1,2,3,4$
k_a $=(I_x/ml^2)^{1/2}$
k_t $=(I_y/ml^2)^{1/2}$
l = reference length
M $=\gamma k_t^{-2} C_{M_\alpha}^*$
M_X, M_Y, M_Z = missile-fixed components of the aerodynamic moment
M_{YC}, M_{ZC} = transverse fixed-plane components of the control moment
m = mass
P $=(I_x/I_y)\phi'$
p,q,r = missile spin, pitch, and yaw rates measured in the missile-fixed system
$\hat{q},\hat{r}$ = missile pitch and yaw rates measured in the fixed-plane system
S = reference area
s = nondimensional arc length along the trajectory
s_g = gyroscopic stability factor $=P^2/4M$
T $=\gamma C_{L_\alpha}^* + \gamma k_a^{-2} C_{M_{p\alpha}}^*$
t = time
u,v,w = missile-fixed components of the velocity
V = magnitude of the velocity
$\hat{x},\hat{y},\hat{z}$ = fixed-plane axes; the $\hat{x}$ axis along the missile's longitudinal axis and the $\hat{y}$ axis always in the horizontal plane
α,β = angles of attack and sideslip in the missile-ixed system
γ $=uV^{-1}$
δ_f = fin cant angle
θ = angle between the missile's axis and the horizontal
θ_T = trajectory angle
λ_j $=K_j'/K_j$, $j=1,2$
μ $=(q+ir)lV^{-1}$
ξ $=(v+iw)V^{-1} \doteq \beta+i\alpha$
ρ = air density
Φ = that part of the fixed-plane complex yaw forcing function due to the spin of the fixed-plane system
ϕ' $=plV^{-1}$
$\dot\phi_{FP}$ = spin rate of the fixed-plane system
ϕ_j = orientation angle of the jth modal arm, $j=1,2$
$\vec{\Omega}_{FP}$ = angular velocity of the fixed-plane system

Superscripts

$(\dot{\ })$ = d( )/dt
$(\ )'$ = d( )/ds = $(\dot{\ })lV^{-1}$
$(\ )^*$ = $(\rho Sl/2m)(\ )$. . . except for g^*
$(\hat{\ })$ = fixed-plane value of ( )
$(\bar{\ })$ = complex conjugate of ( )

Subscripts

e = steady-state equilibrium value
g = steady-state equilibrium value due to gravity

Introduction

IN 1977, Lloyd and Brown[1] investigated the feasibility of controlling a 105 mm spinning projectile by means of horizontal and vertical forces. Their numerical calculations yielded the surprising result that an applied constant-amplitude yaw moment could cause dynamic instability. The usual linear analysis seems to predict that such a moment

Presented as Paper 79-1669 at the AIAA Atmospheric Flight Mechanics Conference, Boulder, Colo., Aug. 6-8, 1979; submitted Nov. 15, 1979; revision received June 9, 1980. This paper is declared a work of the U.S. Government and therefore is in the public domain.

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would cause a steady-state horizontal trim angle but would have no effect on the dynamic stability.

This difficulty was resolved by Lloyd and Brown through the observation that the differential equation for the angular motion in fixed-plane coordinates† contained nonlinear terms in $\dot{\phi}_{FP}$, the coordinate system spin rate. These usually neglected terms vanish completely when the equations are transformed to nonrolling coordinates. The terms involving the horizontal and vertical control moments become nonlinear terms that can be easily linearized. The resulting sixth-order system can be approximately solved for large gyroscopic stability factor ($s_g > 4$) and excellent agreement with the numerical results obtained. The theory, however, only partially considers the influence of gravity and neglects the effect of drag, roll damping moment, and the gradient in air density.

In this paper, we will show that the coordinate system transformation is unnecessary and that a proper linearization in the fixed-plane coordinates requires the solution of a much simpler fourth-order system. This allows the very easy inclusion in the theory of the full effect of gravity as well as the effects of drag, roll damping, and air density gradient. Much more importantly, the requirement of high stability factor is eliminated so that the very significant case of a finned missile with little or no spin can be studied.

Equations of Motion

The first aerodynamic force usually considered is the drag force, which acts along the velocity vector. If we define θ_T as the angle between this vector along the trajectory and the horizontal plane, the velocity equation has the form

$$V'/V = -C_D^* - g^* \tag{1}$$

where

$$C_D^* = (\rho Sl/2m)C_D \qquad g^* = glV^{-2}\sin\theta_T$$

and where the derivative is with respect to the nondimensional arc length s. The aerodynamic moment and transverse force are usually assumed to depend linearly on various angles and angular velocities. If only those linear coefficients are retained that have a measurable effect on the motion, the moment and transverse force can be expressed in missile-fixed coordinates[2] as

$$M_X = \tfrac{1}{2}\rho SlV^2 \left[\delta_f C_{l_\delta} + \phi' C_{l_p}\right] \tag{2}$$

$$M_Y + iM_Z = \tfrac{1}{2}\rho SlV^2 \left[\left(\phi' C_{M_{p\alpha}} - iC_{M_\alpha}\right)\xi + \Phi_{\Omega_\alpha}\mu - iC_{M_{\dot\alpha}}\left(\xi' + i\phi'\xi\right)\right] \tag{3}$$

$$F_Y + iF_Z = -\tfrac{1}{2}\rho SV^2 C_{N_\alpha}\xi \tag{4}$$

where

$$\phi' = plV^{-1} \qquad \xi = (v+iw)V^{-1} \qquad \mu = (q+ir)lV^{-1}$$

The complex variable ξ locates the plane of the velocity vector and has a magnitude that is the sine of the total angle of attack.

The roll equation obtained for the roll moment of Eq. (2) differs from the usual roll equation[2] by a gravity term that acts on the dynamic pressure

$$\phi'' = (D_1 + g^*)\phi' + D_2 \tag{5}$$

where

$$D_1 = C_D^* + k_a^{-2}C_{l_p}^* \qquad D_2 = k_a^{-2}\delta_f C_{l_\delta}^* \qquad k_a = (I_x/ml^2)^{1/2}$$

†Fixed-plane axes $\hat{x}$, $\hat{y}$, $\hat{z}$ pitch and yaw with the missile but roll so that the $\hat{y}$ is always in the horizontal plane.

In addition to the aerodynamic force and moment and the gravity force, we assume constant amplitude control forces and moments that are perpendicular to the projectile's axis and either in the horizontal plane or the vertical plane containing the missile's axis. These control forces, which could be produced by roll-stabilized canards, make fixed-plane coordinates most suitable for the analysis. Since fixed-plane axes pitch and yaw with the projectile but roll so that the $\hat{y}$ axis is always horizontal, it can be shown that this system has an angular velocity vector $\vec{\Omega}_{FP}$ that can be expressed in fixed-plane coordinates[3] as

$$\vec{\Omega}_{FP} = (\dot{\phi}_{FP}, \hat{q}, \hat{r}) \tag{6}$$

$$\dot{\phi}_{FP} = -\hat{r}\tan\theta \tag{7}$$

where θ is the angle between the missile's axis and the horizontal ($\dot{\theta} = \hat{q}$, $\theta \doteq \theta_T + \hat{\alpha}$); and $\hat{q}$, $\hat{r}$ are the pitch and yaw rates in the fixed-plane coordinates.

For this coordinate system, the derivatives of the linear and angular momentum can be computed in the usual way[2,3] and set equal to the sum of the external forces and moments. The equations for the transfer components can then be given in the form of two first-order complex differential equations:

$$\hat\xi' - i\gamma\hat\mu = -i\phi'_{FP}\hat\xi - \gamma C_{L_\alpha}^*\hat\xi + (i\cos\theta + \hat\xi\sin\theta_T)glV^{-2} + (F_{YC} + iF_{ZC})l(mV^2)^{-1} \tag{8}$$

$$\hat\mu' - iP\hat\mu = k_t^{-2}\left[\phi' C_{M_{p\alpha}}^* - iC_{M_\alpha}^*\right]\hat\xi + \left[k_t^{-2}C_{M_q}^* + C_D^* + g^*\right]\hat\mu - ik_t^{-2}C_{M_{\dot\alpha}}^*(\hat\xi' + i\phi'_{FP}\hat\xi) + k_t^{-2}(M_{YC} + iM_{ZC})(mV^2)^{-1} - i\phi'_{FP}\hat\mu \tag{9}$$

where $P = I_x\phi'/I_y$ and γ is the cosine of the total angle of attack (uV^{-1}).

For most missiles, the starred coefficients are of the order 10^{-3} while the dimensionless control forces and moments will be limited to at most 5×10^{-4}. Thus, products of these terms can be neglected when $\hat\mu$ is eliminated between Eqs. (8) and (9). For simplicity, the small geometric nonlinear terms in γ' are neglected and γ is taken to be unity. The resulting second-order complex equation is

$$\hat\xi'' + [H - g^* - iP]\hat\xi' - [M + iPT]\hat\xi = \Phi + \hat{G} + C \tag{10}$$

where

$$\Phi = -i\left\{\left[\phi''_{FP} + (H - g^* - iP)\phi'_{FP} + i(\phi'_{FP})^2\right]\hat\xi + 2\phi'_{FP}\hat\xi'\right\}$$

$$\hat{G} = \left[P(\cos\theta - i\hat\xi\sin\theta_T) + (\hat\xi' + i\phi'_{FP}\hat\xi)(\sin\theta_T - \sin\theta)\right]glV^{-2}$$

$$C = i\left[k_t^{-2}(M_{YC} + iM_{ZC}) - (P - \phi'_{FP})l(F_{YC} + iF_{ZC})\right](mV^2)^{-1}$$

$$\doteq ik_t^{-2}(M_{YC} + iM_{ZC})(mV^2)^{-1}$$

Linearized Equations

Since $\theta = \theta_T + \hat\alpha$ and a good approximation for $\hat{r}$ is $-\dot{\hat\beta}$,

$$\phi'_{FP} \doteq \hat\beta'\tan(\theta_T + \hat\alpha) \tag{11}$$

When this relation for ϕ'_{FP} is substituted in the definition of Φ, we see that Φ is a nonlinear function of the angles and their derivatives. In general, its lowest order term is quadratic. For horizontal flight, however, $\theta_T = 0$ and the lowest order term becomes cubic. For horizontal flight without control moments or gravity, the quasilinear analysis of Ref. 3 shows that the cubic term in Φ causes a change in frequency. Numerical calculations by Clark and Hodapp[4] show that this frequency shift is very well-predicted by the quasilinear analysis.

When $\hat{G}$ is included in the analysis, however, a gravity-induced equilibrium angle $\hat{\xi}_g$ is produced from the constant part of $\hat{G}$. Dynamic stability refers to the decay of motions near this equilibrium angle and thus Φ and $\hat{G}$ should be linearized with respect to $\hat{\xi}_g$. In Ref. 5, it is shown that the linear terms of $\hat{G}$ and the linear terms of Φ associated with $\hat{\xi}_g$ cancel each other's contributions to dynamic stability. For algebraic simplicity, we will completely neglect $\hat{G}$ and $\hat{\xi}_g$, which is usually a quite small angle, and consider only the control moment C, its equilibrium angle $\hat{\xi}_e$, and the linear terms of Φ associated with this control-moment-induced trim angle.

According to Eq. (10), equilibrium is obtained for

$$\hat{\xi}=\hat{\xi}_e=-C/(M+iPT)=\hat{\beta}_e+i\hat{\alpha}_e \tag{12}$$

$$\theta_e=\theta_T+\hat{\alpha}_e \tag{13}$$

Φ can now be expanded in terms of $\hat{\xi}-\hat{\xi}_e$ and its derivatives to yield the following linear part:

$$\Phi \doteq -2iE[\hat{\beta}''+(H-g^*-iP)\hat{\beta}'] \tag{14}$$

where

$$E=\tfrac{1}{2}\hat{\xi}_e\tan\theta_e$$

$\hat{\beta}$ in Eq. (14) can be related to the complex variable $\hat{\xi}$ by $\hat{\beta}=\tfrac{1}{2}(\hat{\xi}+\bar{\hat{\xi}})$ and the result used to reduce Eq. (10) to the following form:

$$(1+iE)[\hat{\xi}''+(H-g^*-iP)\hat{\xi}']-(M+iPT)(\hat{\xi}-\hat{\xi}_e)=-iEf \tag{15}$$

$$f=\bar{\hat{\xi}}''+(H-g^*-iP)\bar{\hat{\xi}}' \tag{16}$$

Quasisymmetric Solution

In Ref. 6, the angular motion of asymmetric missiles was considered and it was shown that the effect of the asymmetries was to add terms in $\bar{\xi}$ and $\bar{\xi}'$ to the simple second-order equation in ξ for the angular motion of symmetric missiles. The motion of a symmetric missile without trim can be described by a two-mode epicyclic motion while the motion with trim requires a three-mode tricyclic motion. The inclusion of the conjugate terms required the addition of two more modes:

$$\hat{\xi}-\hat{\xi}_e=K_1e^{i\phi_1}+K_2e^{i\phi_2}+K_3e^{i\phi_3}+K_4e^{i\phi_4} \tag{17}$$

where

$$\phi_3'=-\phi_1' \qquad \phi_4'=-\phi_2'$$

$$K_j=K_{j0}e^{\lambda_1 s} \qquad (j=1,3)$$

$$=K_{j0}e^{\lambda_2 s} \qquad (j=2,4)$$

For nonzero or nonresonant spin, it was shown that the amplitudes of the additional modes decayed as the asymmetry went to zero. A good approximation for small asymmetries was obtained by neglecting K_3 and K_4. This was equivalent to neglecting the terms in $\bar{\xi}$ and $\bar{\xi}'$.

In a similar way we will obtain a good approximate solution for Eq. (15) by neglecting Ef and using Eq. (17) with $K_3=K_4=0$. For small E and the usual size assumptions ($|\lambda_j| \ll |\phi_j'|$, $|H| \ll |\phi_j'|$, $|T| \ll |\phi_j'|$), equations for the two frequencies and damping rates can be obtained for $s_g>1$ or $s_g<0$:

$$(\phi_j')^2-P\phi_j'+M+\tan\theta_e[M\hat{\alpha}_e+PT\hat{\beta}_e]/2=0 \tag{18}$$

$$\lambda_j=\frac{-(H-g^*)\phi_j'+PT-\phi_j''-\tan\theta_e[M\hat{\beta}_e-PT\hat{\alpha}_e]/2}{2\phi_j'-P} \tag{19}$$

H contains the drag coefficient, but an additional effect of the drag on damping appears in ϕ_j''. This term also contains the influence of spin decay and the air density gradient. A good estimate for ϕ_j'' can be obtained from Eq. (18) by neglecting the small trim angle terms and differentiating:

$$\phi_j''=(\phi_j'P'-M')/(2\phi_j'-P) \tag{20}$$

P' is given by Eq. (5) and M' for constant C_{M_α} is $(\rho'/\rho)M$.

For near zero spin or near resonant spin, it may not be valid to assume that K_3 and K_4 can be neglected. In Ref. 7, however, it is shown that this assumption is valid for spin rates as low as five times resonance spin ($\phi_{\text{reson}}'=[-M]^{1/2}$) and Eqs. (18) and (19) are correct for these spin rates. For lower spins, numerical integration of the differential equations should be used to determine the complete effect of a control moment on the angular motion.

For zero spin, exact solutions to the equations of motion can be obtained:

$$\hat{\xi}=\hat{\xi}_e+K_1e^{i\phi_1}+K_2e^{i\phi_2}+F \tag{21}$$

where

$$F=-K_1e^{-i\phi_1}-K_2(\hat{\xi}_e/\bar{\hat{\xi}}_e)e^{-i\phi_2} \qquad \hat{\alpha}_e\neq 0$$

$$=i[-M]^{1/2}E[K_1\sin\phi_1-K_2\sin\phi_2]s \qquad \hat{\alpha}_e=0$$

$$K_j=K_{j0}\ \exp\{-(H-g^*)s\}$$

$$\phi_1'=[-M]^{1/2} \qquad \phi_2'=-[-(1+\hat{\alpha}_e\tan\theta_e)M]^{1/2}$$

Equation (21) can be verified by direct substitution in Eq. (15). The presence of a secular term in F for $\hat{\alpha}_e=0$ shows the singular behavior near zero spin and the need for numerical simulations.

Discussion

Since $|PT|$ is usually much smaller than $|M|$, the effect of $\hat{\alpha}_e$ on the damping rates can be neglected. Equation (19) can therefore be used to derive stability boundaries for the maximum trim angles for $\hat{\beta}$. For a gyroscopically stable missile with positive spin ($\phi_1'>P/2>\phi_2'>0$),

$$B_1<\hat{\beta}_e\tan\theta_e<B_2 \tag{22}$$

where

$$B_j=-(2/M)[(H-g^*)\phi_j'-PT+\phi_j'']$$

Table 1 gives the various parameters for a 105 mm shell and Fig. 1 gives the boundaries B_1 and B_2 in degrees as functions of the gyroscopic stability factor s_g. The Lloyd and Brown results were limited to large stability factors and their numerical calculations were for $s_g=5.8$. As we can see from

Table 1 Assumed parameters for 105 mm shell

$l=0.105$ m	$C_D=0.13$
$S=0.0087$ m^2	$C_{L_\alpha}=1.7$
$m=15$ kg	$C_{l_p}=-0.012$
$I_x=0.023$ kg-m^2	$C_{M_\alpha}=3.8$
$I_y=0.22$ kg-m^2	$C_{M_{p\alpha}}=0.2$
$\rho=(1.22\ \text{kg/m}^3)\exp(-\sigma z)$	$C_{M_q}+C_{M_{\dot{\alpha}}}=-8$
$z=1000$ m	$C_{l_\delta}=0$
$V=250$ m/s	
$p=1050$ rad/s	
$\sigma=(6700\ \text{m})^{-1}$	

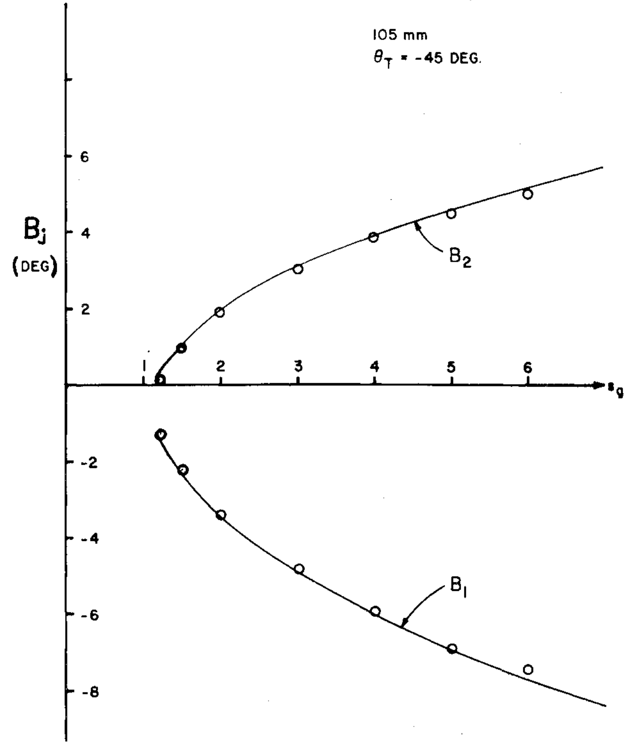

Fig. 1 Stability boundaries on the trim term $\hat{\beta}_e \tan\theta_e$ for the 105 mm shell descending from 1000 m.

Fig. 1, the allowable range for β_e becomes quite small for stability factors less than 2.

Exact numerical integrations of Eq. (15) have been made for s_g as low as 1.2 and the stability boundary for $\hat{\beta}_e$ determined. The numerically obtained boundary values are indicated by open circles in Fig. 1. As can be seen from the figure, the agreement with theory is excellent. For s_g closer to unity than 1.2, ϕ_j' is very near to $P/2$, and λ_j as given by Eq. (19) is no longer small compared to ϕ_j'. Equations (18-20) are no longer valid and a more complicated derivation is required.

Conclusions

1) The Lloyd-Brown control-moment-induced instability can be predicted by linearization of the fixed-plane equations with respect to the trim angle.

2) The resulting stability criterion correctly includes the effects of velocity variation due to drag and gravity plus the effects of air density gradient and spin variation. This criterion is valid for missiles with low gyroscopic stability as well as for statically stable missiles with spins well above resonance spin.

3) Angular motion of slower spinning, statically stable missiles with control moments should be studied by numerical simulation.

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